

The momentum of an electromagnetic wave inside a dielectric

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Abstract

The problem of assigning a momentum to an electromagnetic wave packet propagating inside an insulator has become known under the name of the Abraham-Minkowski controversy. In the present paper we re-examine the question, first through a power expansion in the polarizability of the medium and assuming the simplest and most natural choice for the force exerted on a dielectric material by an electromagnetic field. It is shown that the Abraham expression is highly favoured. We then show the complete generality of these results.

1 Introduction

It is well known that an electromagnetic (e.m.) wave propagating in the vacuum carries both energy and momentum[1],[2]. Of course this must be true also for radiation propagating inside matter. Soon after the formulation of Einstein Relativity Theory, conflicting expressions were proposed for the space momentum to be assigned in this situation[3],[4]. This discrepancy, known as the Abraham-Minkowski controversy, is still attracting a considerable interest in part of the scientific community.²

In order to discuss the problem we imagine an idealized situation in which matter is schematized as a uniform, isotropic dielectric material

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²For a review of the subject see[5],[6] and [7].

\mathcal{D} , occupying the space region $V_{\mathcal{D}}$, with dielectric constant ϵ , refraction index $n = \sqrt{\epsilon}$, with no magnetic properties, i.e. with permeability $\mu = 1$ and sufficiently massive that it can transfer space momentum, but not energy with the incident radiation. The dielectric is surrounded by empty space, $V_{\mathcal{V}}$. If stress variations induced by e.m. radiation can be safely neglected, the Abraham-Minkowski discrepancy can be formulated imagining a collimated e.m. wave packet, coming from the vacuum, \mathcal{V} , with an energy ω and a momentum \mathbf{p} ($\omega = c|\mathbf{p}|$), entering the dielectric material. Once this radiation is fully inside the dielectric

- Minkowski attributes to it the same vacuum energy ω and an absolute value of the momentum $n|\mathbf{p}|$,
- while Abraham attributes to it again the same ω , but and an absolute value of the momentum $\frac{|\mathbf{p}|}{n}$.

It is the purpose of the present paper to give a small contribution to this debate through the discussion of examples, amenable to quantitative conclusions, which clearly favour the Abraham point of view on the momentum. Approximate computations are performed in sections 3.1, 3.2, 3.3, under the assumption that the dielectric polarizability of the medium is small. In section 4 the problem is discussed without approximations, studying the transition of the e.m. field through the dielectric boundary in full generality.

2 Polarization charges and currents

The description of a transparent, dielectric, homogeneous material with dielectric constant ϵ and magnetic permeability $\mu = 1$ requires the introduction of the polarization field, $\mathbf{P}(\mathbf{x}, t)$, in terms of which the polarization charge density $\rho_P(\mathbf{x}, t)$ is given as

$$\rho_P(\mathbf{x}, t) = -\nabla \cdot \mathbf{P}(\mathbf{x}, t). \quad (1)$$

The time dependence of the polarization field gives rise, as a consequence of charge conservation, to a volume current density described by

$$\mathbf{J}_P(\mathbf{x}, t) = \dot{\mathbf{P}}(\mathbf{x}, t). \quad (2)$$

The Maxwell equations in presence of polarization are written as

$$\nabla \cdot \mathbf{E} = -\nabla \cdot \mathbf{P} \quad \nabla \cdot \mathbf{B} = 0 \quad (3)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \dot{\mathbf{B}} \quad \nabla \times \mathbf{B} = \frac{1}{c} \dot{\mathbf{E}} + \frac{1}{c} \dot{\mathbf{P}}. \quad (4)$$

In the linear regime, $\mathbf{P}(\mathbf{x}, t)$ is related to the electric field $\mathbf{E}(\mathbf{x}, t)$ through

$$\mathbf{P}(\mathbf{x}, t) = (\epsilon - 1) \mathbf{E}(\mathbf{x}, t) \equiv \alpha \mathbf{E}(\mathbf{x}, t), \quad (5)$$

which defines the polarizability α .

In the following we will consider a uniform dielectric, i.e. we will consider the polarizability α as constant, except in a thin region around the dielectric boundary, in which it quickly drops from α to 0. Whenever possible, without giving rise to any ambiguity, we will treat this transition region according to distribution theory[13] and will schematize the situation through the introduction of a surface density $\tilde{\sigma}_P$ of polarization charges induced on the dielectric surface Σ

$$\rho_P(\mathbf{x}, t) = -\nabla \cdot \mathbf{P}(\mathbf{x}, t) = \alpha (\mathbf{n}_D \cdot \mathbf{E}(\mathbf{x}, t)) \delta(\Sigma(\mathbf{r})) \equiv \tilde{\sigma}_P(\mathbf{x}, t) \delta(\Sigma(\mathbf{r})), \quad (6)$$

where \mathbf{n}_D is the outward normal to the dielectric surface, $\Sigma(\mathbf{r}) = 0$, and $\delta(\Sigma(\mathbf{r}))$ is the surface Dirac δ -function[13] with support on Σ . In eq.(6) we defined

$$\tilde{\sigma}_P(\mathbf{x}, t) \equiv \alpha (\mathbf{n}_D \cdot \mathbf{E}(\mathbf{x}, t)). \quad (7)$$

The electric field $\mathbf{E}(\mathbf{x}, t)$ appearing in eqs.(6) and (7) is thought to be obtained reaching the surface Σ from the dielectric side.

According to eq.(6), polarization charges are not present in the dielectric interior

$$\begin{aligned}\rho_P(\mathbf{x}, t) &= -\nabla \cdot \mathbf{P}(\mathbf{x}, t) = -\alpha \nabla \cdot \mathbf{E}(\mathbf{x}, t) = 0, \\ \mathbf{x} &\notin \Sigma,\end{aligned}\tag{8}$$

in virtue of the hypothesis that the sources generating the incident electromagnetic field are external to the dielectric[9].

We remind the reader that across the dielectric boundary we have the following jump conditions[1]

$$\Delta E_n = \tilde{\sigma}_P \quad \Delta \mathbf{E}_{||} = 0\tag{9}$$

$$\Delta \mathbf{B} = 0,\tag{10}$$

where E_n and $\mathbf{E}_{||}$ denote the normal and tangential components of the electric field on Σ and the symbol Δ denotes the increment in passing from \mathcal{D} to \mathcal{V} .

3 The forces on a dielectric material and the α -expansion

One fundamental issue raised in the literature, concerns the precise form of the e.m. force acting on a material, in terms of the matter polarization[8],[9],[10],[11],[12].

We will take the simplest possible attitude, assuming that the e.m. field acts on the polarization charges and currents in the same way as it acts on the so called *free* charges and currents[14]. Moreover, as discussed in the introduction, we neglect forces due to a change of the stress status of the dielectric body. The total electromagnetic force acting on the dielectric, occupying the region $V_{\mathcal{D}}$, is therefore computed as

$$\mathbf{F}_{\mathcal{D}}(t) = \int_{V_{\mathcal{D}}} \rho_P(\mathbf{x}, t) \mathbf{E}(\mathbf{x}, t) d\mathbf{r} + \frac{1}{c} \int_{V_{\mathcal{D}}} \mathbf{J}_P(\mathbf{x}, t) \times \mathbf{B}(\mathbf{x}, t) d\mathbf{r} = \tag{11}$$

$$= - \int_{V_D} [\mathbf{E} \cdot \nabla \alpha] \mathbf{E} d\mathbf{r} + \frac{\alpha}{c} \int_{V_D} \dot{\mathbf{E}} \times \mathbf{B} d\mathbf{r}. \quad (12)$$

If we consider the transition from the dielectric to the vacuum as sharp, i.e. if we insist using eq.(6), we must consider eqs.(11) and (12) as formal. In fact $\rho_P(\mathbf{x}, t)$ is a distribution singular on Σ , as shown in eq.(6) or, equivalently, the term $\nabla \alpha$ appearing in eq.(12), is a distribution singular on Σ . The problem is that in eqs.(11) and (12) these distributions appear multiplied by \mathbf{E} , which is itself a distribution singular³ on Σ . This is an illegal operation in distribution theory and could give rise to ambiguities or infinities. In the present case, for example, it is not clear if the $\mathbf{E}(\mathbf{x}, t)$ appearing in eqs.(11) and (12) should be taken from the vacuum or the insulator side.

We will not need, at the moment, the full solution to this problem, which we postpone to section 4. In fact, in order to be able to perform computations, we will introduce an approximation scheme. The tool we will use is to imagine an homogeneous, *weak* dielectric, namely a material with a polarizability such that

$$\alpha \equiv \epsilon - 1 \approx 0. \quad (13)$$

Eq.(13) allows us to consider an expansion in powers of α , first used by J. P. Gordon[15]. We will apply this approximation scheme up to first order in α , which is easy and accurate enough to distinguish between the Abraham or Minkowski formulation and is not plagued by any ambiguity, as it will be clear in a moment.

We consider therefore a dielectric block invested by an e.m. wave described by a given electric field $\mathbf{E}_0(\mathbf{x}, t)$ and a given magnetic field $\mathbf{B}_0(\mathbf{x}, t)$, generated by sources external to the dielectric itself.

Since we deal with infinitely massive matter, it is not possible to trace directly the momentum flow coming from radiation. We can,

³ \mathbf{E} is discontinuous across Σ , see eq.(9).

instead, identify the transferred momentum through the knowledge of the force that the e.m. wave exerts on the dielectric[16],[17].

Up to order α , the force acting on the dielectric \mathcal{D} , formally given by eq.(12), reads

$$\begin{aligned}\mathbf{F}_{\mathcal{D}}(t) &\approx - \int_{V_{\mathcal{D}}} [\mathbf{E}_0 \cdot \nabla \alpha] \mathbf{E}_0 d\mathbf{r} + \frac{\alpha}{c} \int_{V_{\mathcal{D}}} \dot{\mathbf{E}}_0 \times \mathbf{B}_0 d\mathbf{r} = \\ &= \alpha \int_{\Sigma} (\mathbf{E}_0 \cdot \mathbf{n}_{\mathcal{D}}) \mathbf{E}_0 d\Sigma + \frac{\alpha}{c} \int_{V_{\mathcal{D}}} \dot{\mathbf{E}}_0 \times \mathbf{B}_0 d\mathbf{r},\end{aligned}\quad (14)$$

which shows that, up to order α , no ambiguity is present, since \mathbf{E}_0 is perfectly regular on Σ .

The general treatment of the singularity problem, present in eqs.(11) and (12) will be given in section 4.

The use the identity

$$\frac{1}{c} \mathbf{E}_0 \times \dot{\mathbf{B}}_0 = (\mathbf{E}_0 \cdot \nabla) \mathbf{E}_0 - \frac{1}{2} \nabla (\mathbf{E}_0^2), \quad (15)$$

which follows from the source-free Maxwell equations, allows us to complete the time derivative in eq.(14) and write $\mathbf{F}_{\mathcal{D}}(t)$ as

$$\mathbf{F}_{\mathcal{D}}(t) \approx \frac{\alpha}{c} \frac{d}{dt} \int_{V_{\mathcal{D}}} \mathbf{E}_0 \times \mathbf{B}_0 d\mathbf{r} + \frac{\alpha}{2} \int_{V_{\mathcal{D}}} \nabla (\mathbf{E}_0^2) d\mathbf{r} = \quad (16)$$

$$= \frac{\alpha}{c} \frac{d}{dt} \int_{V_{\mathcal{D}}} \mathbf{E}_0 \times \mathbf{B}_0 d\mathbf{r} + \frac{\alpha}{2} \int_{\Sigma} \mathbf{n}_{\mathcal{D}} \mathbf{E}_0^2 d\Sigma. \quad (17)$$

In the following sections we will use eqs.(16) and (17) to analyze simple situations, obtaining results in consistent agreement with the Abraham form of the momentum.

3.1 The Einstein box

In a beautiful paper, Balasz[18] considered a thought experiment which consists in sending an e.m. wave packet on a dielectric box. The input used by Balasz in order to predict the outcome, is the requirement that the combined center of mass of the box and of the radiation should

maintain, during the experiment, a uniform, rectilinear motion. On this basis Balasz concluded that, while the radiation penetrates into its interior, the box should be subject to a force in the direction of the e.m. wave, it should then not be subject to any force while the wave is traveling inside the dielectric and should finally be subject to an equal and opposite force, during the time in which the radiation exits from the box. We will analyze this problem from the point of view of eq.(16), finding results in agreement with Balasz analysis and the Abraham form of the momentum⁴.

We consider an unperturbed e.m. wave packet propagating along the positive z -axis, of the form⁵

$$\mathbf{E}_0(\mathbf{x}, t) \equiv (f(z - ct), 0, 0) \quad (18)$$

$$\mathbf{B}_0(\mathbf{x}, t) \equiv (0, f(z - ct), 0), \quad (19)$$

where the function f has a finite support

$$f(z - ct) \neq 0, \quad a \leq z - ct \leq b. \quad (20)$$

Imagining $V_{\mathcal{D}}$ as a cube of side L , with two faces orthogonal to the z -axis, we have, from eq.(16)

$$\begin{aligned} F_{\mathcal{D}z}(t) &\approx \frac{\alpha}{c} \frac{d}{dt} \int_{V_{\mathcal{D}}} g(z - ct) dV + \frac{\alpha}{2} \int_{V_{\mathcal{D}}} \frac{\partial}{\partial z} g(z - ct) dV = \\ &= -\frac{\alpha}{2} \int_{V_{\mathcal{D}}} g'(z - ct) dV, \end{aligned}$$

where $g(z - ct) \equiv f^2(z - ct)$, so that

$$F_{\mathcal{D}z}(t) \approx \alpha \frac{A}{2} [g(-ct) - g(L - ct)], \quad (21)$$

where A , smaller than L^2 , is the transverse section of the wave packet. If in eq.(21) we take $L > b - a$, then $g(-ct)$ and $g(L - ct)$ cannot

⁴In this section we follow ref.[15]

⁵In the spirit of the α -expansion, $\mathbf{E}_0(\mathbf{x}, t)$ and $\mathbf{B}_0(\mathbf{x}, t)$ are the electric and magnetic fields at zeroth order, α^0 , and therefore they freely propagate at the speed of light in empty space, for all t .

be both different from 0 for any t and we can consider three distinct regions of time:

- Region I

$-\frac{b}{c} \leq t \leq -\frac{a}{c}$, during which the e.m. wave is entering in $V_{\mathcal{D}}$ and a positive force is exerted on the dielectric,

- Region II

$-\frac{a}{c} \leq t \leq \frac{L-b}{c}$, during which the e.m. wave is traveling inside $V_{\mathcal{D}}$ and no force is exerted on the dielectric and

- Region III

$\frac{L-b}{c} \leq t \leq \frac{L-a}{c}$, during which the e.m. wave is exiting from $V_{\mathcal{D}}$ and a negative force is exerted on the dielectric,

in agreement with the Balasz analysis.

3.2 The e.m. momentum inside the Einstein box

In the Einstein box experiment discussed in section 3.1, the momentum of the incident e.m. wave packet is directed along the z -axis and its z -component is given by

$$P_{\gamma}^{(0)} = \frac{1}{c} \int_{V_{\mathcal{D}}} g(z - ct) dV = \frac{A}{c} \int_a^b g(z) dz. \quad (22)$$

During the time region I, the momentum transferred to the dielectric block through e.m. forces is also directed along the z direction and is given by

$$P_{\mathcal{D}}^{(I)} = \int_{-\frac{b}{c}}^{-\frac{a}{c}} F_{\mathcal{D}z}(t) dt = \alpha \frac{A}{2} \int_{-\frac{b}{c}}^{-\frac{a}{c}} g(-ct) dt = \frac{\alpha}{2} P_{\gamma}^{(0)}. \quad (23)$$

Therefore, if we require momentum conservation, we are led to attribute to the e.m. wave, while it propagates inside the dielectric during the

time region II and not exerting any force on it, a z -component of the momentum such that

$$P_{\gamma}^{(II)} = P_{\gamma}^{(0)} - P_{\mathcal{D}}^{(I)} = P_{\gamma}^{(0)}(1 - \frac{\alpha}{2}) \approx \frac{P_{\gamma}^{(0)}}{n}. \quad (24)$$

In eq.(24) we used the relation

$$n = \sqrt{\epsilon} \approx 1 + \frac{\alpha}{2}. \quad (25)$$

relating the refraction index n to the dielectric constant and we recovered the Abraham form of the radiation momentum.

3.3 The case of oblique incidence

In this section we discuss, on the grounds of a more general kinematic configuration, the validity, up to order α , of the Abraham form of the radiation momentum. For this purpose we consider an e.m. wave packet incident on a dielectric boundary in an oblique direction. The dielectric occupies the half space $z \geq 0$, always denoted by $V_{\mathcal{D}}$, and, being as before infinitely massive, it does not exchange energy with the e.m. wave hitting it⁶. The interest of this more general kinematical situation resides in the fact that it allows to recover, besides the Abraham form of the momentum, also the Snell's law for refracted light.

We start integrating eq.(17) over time, from $-\infty$ to $+\infty$ ⁷. We get

$$\begin{aligned} \mathbf{P}_{\mathcal{D}} &= \int_{-\infty}^{+\infty} \mathbf{F}_{\mathcal{D}}(t) dt \approx \\ &\approx \frac{\alpha}{c} \left(\int_{V_{\mathcal{D}}} \mathbf{E}_0 \times \mathbf{B}_0|_{t=+\infty} d\mathbf{r} - \int_{V_{\mathcal{D}}} \mathbf{E}_0 \times \mathbf{B}_0|_{t=-\infty} d\mathbf{r} \right) + \\ &\quad + \mathbf{n}_{\mathcal{D}} \frac{\alpha}{2} \int_{-\infty}^{+\infty} dt \int_{\Sigma} \mathbf{E}_0^2 d\Sigma = \\ &= \frac{\alpha}{c} \int_{V_{\mathcal{D}}} \mathbf{E}_0 \times \mathbf{B}_0|_{t=+\infty} d\mathbf{r} - \hat{\mathbf{z}} \frac{\alpha}{2} \int_{-\infty}^{+\infty} dt \int_{\Sigma} \mathbf{E}_0^2 d\Sigma \end{aligned} \quad (26)$$

⁶This point will be further discussed in section 4

⁷The t -integration is actually limited to the time interval during which the wave packet crosses the boundary $z = 0$.

where $\mathbf{P}_{\mathcal{D}}$ is the momentum transferred to the dielectric by the radiation, Σ is the dielectric surface, $z = 0$, $\mathbf{n}_{\mathcal{D}}$ its external normal and $\hat{\mathbf{z}} = -\mathbf{n}_{\mathcal{D}}$, the unit vector along the positive z -axis. Since $V_{\mathcal{D}}$ is the region occupied by the dielectric, the term $\int_{V_{\mathcal{D}}} \mathbf{E}_0 \times \mathbf{B}_0|_{t=-\infty} d\mathbf{r}$ is zero, because, for sufficiently negative t , the radiation is supposed to be confined in the region $z < 0$. Similarly we have

$$\frac{1}{c} \int_{V_{\mathcal{D}}} \mathbf{E}_0 \times \mathbf{B}_0|_{t=+\infty} d\mathbf{r} = \mathbf{P}_{\gamma}^{(0)}, \quad (27)$$

where $\mathbf{P}_{\gamma}^{(0)}$ is the *initial* e.m. momentum. In fact as the time evolves we imagine that the unperturbed e.m. radiation will, eventually, be entirely inside $V_{\mathcal{D}}$: since the electric and magnetic fields $\mathbf{E}_0(\mathbf{x}, t)$, $\mathbf{B}_0(\mathbf{x}, t)$ propagate freely, as explained in the note 5, the integral appearing in eq.(27) coincides with the initial, vacuum momentum of the unperturbed e.m. field.

Summing up we have

$$\mathbf{P}_{\mathcal{D}} = \alpha \mathbf{P}_{\gamma}^{(0)} - \hat{\mathbf{z}} \frac{\alpha}{2} \int_{-\infty}^{+\infty} dt \int_{\Sigma} \mathbf{E}_0^2 d\Sigma. \quad (28)$$

In order to give an estimate of the last term in eq.(28), we use the approximation that the wave packet propagates without deformation, while crossing the dielectric boundary. In other words we assume

$$\mathbf{E}_0^2(\mathbf{x}, t) = D(\mathbf{x} - ct \mathbf{k}), \quad (29)$$

where \mathbf{k} is the unit vector which defines the (incident) e.m. wave direction.

Eq.(29) is valid for wave packets containing short wave lengths and propagating over a not too long time-lapse and it will be a good approximation for e.m. wave packets with small space extension.

Performing the change of integration variables $\mathbf{y} = \mathbf{x} - ct \mathbf{k}$, we easily get

$$\int_{-\infty}^{+\infty} dt \int_{\Sigma} \mathbf{E}_0^2 d\Sigma = \int_{-\infty}^{+\infty} dt \int_{\Sigma} D(\mathbf{x} - ct \mathbf{k}) d\Sigma =$$

$$= \frac{1}{k_z c} \int_{-\infty}^{+\infty} d\mathbf{y} D(\mathbf{y}) = \frac{U_\gamma^{(0)}}{ck_z} = \frac{P_\gamma^{(0)}}{k_z}, \quad (30)$$

where $U_\gamma^{(0)} = cP_\gamma^{(0)}$ is the e.m. energy of the free initial wave packet and

$$k_z = \cos \hat{i}, \quad (31)$$

\hat{i} being the angle between the direction of the incident e.m. wave and the z -axis. Therefore we obtain, for the total momentum transferred to the dielectric wall,

$$\mathbf{P}_D = \alpha \mathbf{P}_\gamma^{(0)} - \hat{\mathbf{z}} \frac{\alpha}{2 \cos \hat{i}} P_\gamma^{(0)}. \quad (32)$$

By the requirement of momentum conservation, we are therefore led to attribute to the e.m. wave, while inside the dielectric, a momentum \mathbf{P}'_γ such that

$$\mathbf{P}_\gamma^{(0)} = \mathbf{P}'_\gamma + \mathbf{P}_D = \mathbf{P}'_\gamma + \alpha \mathbf{P}_\gamma^{(0)} - \hat{\mathbf{z}} \frac{\alpha}{2 \cos \hat{i}} P_\gamma^{(0)}, \quad (33)$$

so that

$$\mathbf{P}'_\gamma = (1 - \alpha) \mathbf{P}_\gamma^{(0)} + \hat{\mathbf{z}} \frac{\alpha}{2 \cos \hat{i}} P_\gamma^{(0)}. \quad (34)$$

Let us consider the implications eq.(34). First of all we have, to order α ,

$$\begin{aligned} (\mathbf{P}'_\gamma)^2 &= (1 - 2\alpha) (\mathbf{P}_\gamma^{(0)})^2 + \frac{\alpha}{\cos \hat{i}} P_\gamma^{(0)} (P_\gamma^{(0)})_z = \\ &= (1 - 2\alpha) (\mathbf{P}_\gamma^{(0)})^2 + \alpha (\mathbf{P}_\gamma^{(0)})^2 = (1 - \alpha) (\mathbf{P}_\gamma^{(0)})^2 = \frac{1}{n^2} (\mathbf{P}_\gamma^{(0)})^2, \end{aligned} \quad (35)$$

which again confirms the Abraham expression for the radiation momentum. Moreover, from eq.(34), we also get

$$\begin{aligned} \sin \hat{i}' &= \frac{(\mathbf{P}'_\gamma)_\perp}{P'_\gamma} = \frac{(1 - \alpha)(\mathbf{P}_\gamma^{(0)})_\perp}{(1 - \frac{\alpha}{2})(P_\gamma^{(0)})} = \\ &= (1 - \frac{\alpha}{2}) \frac{(\mathbf{P}_\gamma^{(0)})_\perp}{(P_\gamma^{(0)})} = \frac{1}{n} \frac{(\mathbf{P}_\gamma^{(0)})_\perp}{(P_\gamma^{(0)})} = \frac{1}{n} \sin \hat{i}, \end{aligned} \quad (36)$$

where \hat{i}' is the refraction angle. Eq.(36) shows the validity of Snell's law, starting from the conservation of the Abraham momentum⁸.

4 Exact results

Previous sections suggest that, up to order α , once the e.m. wave crosses the dielectric boundary, its associated momentum changes from the initial momentum \mathbf{p} the wave possessed in the vacuum, to the Abraham momentum value $\frac{\mathbf{p}}{n}$, inside the dielectric. This has been checked indirectly, assuming momentum conservation and attributing to the e.m. wave the missing momentum. The physical picture associated with the change in momentum of the e.m. wave, is that the incident e.m. radiation, once it reaches the dielectric boundary, excites polarization charges and currents which induce an e.m. field carrying the additional missing momentum.

In order to study this problem we should approximate the e.m field up to order α

$$\mathbf{E} \approx \mathbf{E}_0 + \mathbf{E}_1 \quad (37)$$

$$\mathbf{B} \approx \mathbf{B}_0 + \mathbf{B}_1. \quad (38)$$

This can be done[19] and also the Abraham Poynting vector

$$\mathbf{\Pi} = \frac{1}{c} \mathbf{E} \times \mathbf{B}, \quad (39)$$

should be expanded according to

$$\mathbf{\Pi} \approx \mathbf{\Pi}_0 + \mathbf{\Pi}_1, \quad (40)$$

where

$$\mathbf{\Pi}_0 = \frac{1}{c} \mathbf{E}_0 \times \mathbf{B}_0, \quad (41)$$

⁸In section 4 we show that, up to order α , there is no reflection.

and

$$\mathbf{\Pi}_1 = \frac{1}{c} \mathbf{E}_1 \times \mathbf{B}_0 + \frac{1}{c} \mathbf{E}_0 \times \mathbf{B}_1. \quad (42)$$

Eq.(42) shows that, up to order α , there cannot be any reflected wave: the support of $\mathbf{\Pi}_1$ is contained in that of $\mathbf{\Pi}_0$ and therefore the support of the momentum density $\mathbf{\Pi}$, coincides with the support of the zeroth order e.m. field, \mathbf{E}_0 and \mathbf{B}_0 at all times.

Although the small α -approximation is very useful to get a semi quantitative idea of the basic phenomena related to dielectric polarization, it is not easy to establish the degree of generality of the results obtained in this way.

In this section we will overcome the limitations due to the α -expansion and discuss in general terms the passage of an e.m. wave packet from vacuum to an insulator.

Following Abraham⁹ we make the hypothesis that the e.m. wave momentum is given in any situation¹⁰ by $\int_{\mathcal{R}_\infty} \frac{1}{c} \mathbf{E} \times \mathbf{B} d\mathbf{r}$, where \mathcal{R}_∞ denotes full three dimensional space and we will trace the momentum change while e.m. wave crosses the boundary between the empty region V_V and that occupied by the dielectric, V_D . In this way we will be able to give a meaning to the formal expression for the force exerted on \mathcal{D} given in eqs.(11) and (12), obtaining an unambiguous result, which coincides with that found in textbooks[20] in similar situations.

In order to deal with problems related to the discontinuity of the electric field across Σ , we introduce a regularization. In other words we isolate the surface Σ enclosing it inside a thin layer \mathcal{L}_δ with thickness δ and will eventually take the limit $\delta \rightarrow 0$, in which \mathcal{L}_δ tends to Σ .

We denote by $\mathcal{R}_\infty \ominus \mathcal{L}_\delta$ the full three dimensional space \mathcal{R}_∞ , deprived of \mathcal{L}_δ and consider the quantity $\int_{\mathcal{R}_\infty \ominus \mathcal{L}_\delta} \frac{1}{c} \mathbf{E} \times \mathbf{B} d\mathbf{r}$. If the integration region were \mathcal{R}_∞ , it would represent the Abraham momentum

⁹It is easy to use the same procedure with the Minkowski momentum.

¹⁰We are considering configurations in which the e.m. field is confined in a finite region.

of the e.m. wave. It is clear, at least from physical intuition, that

$$\lim_{\delta \rightarrow 0} \int_{\mathcal{L}_\delta} \frac{1}{c} \mathbf{E} \times \mathbf{B} d\mathbf{r} = 0; \quad (43)$$

in other words we do not expect a finite fraction of the e.m. momentum to be trapped and accumulate inside \mathcal{L}_δ with a vanishing δ . This means that $\lim_{\delta \rightarrow 0} \int_{\mathcal{R}_\infty \ominus \mathcal{L}_\delta} \frac{1}{c} \mathbf{E} \times \mathbf{B} d\mathbf{r}$ correctly gives the Abraham momentum of the e.m. field.

We are now in the position to compute

$$\begin{aligned} & \frac{d}{dt} \int_{\mathcal{R}_\infty \ominus \mathcal{L}_\delta} \frac{1}{c} \mathbf{E} \times \mathbf{B} d\mathbf{r} = \\ & = \int_{\mathcal{R}_\infty \ominus \mathcal{L}_\delta} \frac{1}{c} \dot{\mathbf{E}} \times \mathbf{B} d\mathbf{r} + \int_{\mathcal{R}_\infty \ominus \mathcal{L}_\delta} \frac{1}{c} \mathbf{E} \times \dot{\mathbf{B}} d\mathbf{r}, \end{aligned} \quad (44)$$

where \mathbf{E} and \mathbf{B} are perfectly regular, having isolated the regions containing their discontinuities.

Through the use of the Maxwell eqs.(3) and (4) we transform some of the volume integrals into surface integrals and get

$$\begin{aligned} & \lim_{\delta \rightarrow 0} \frac{d}{dt} \int_{\mathcal{R}_\infty \ominus \mathcal{L}_\delta} \frac{1}{c} \mathbf{E} \times \mathbf{B} d\mathbf{r} = \\ & = \int_{\Sigma} \left[-\frac{1}{2} \mathbf{n}_{\mathcal{D}} \mathbf{E}_{\mathcal{D}}^2 + [(\mathbf{n}_{\mathcal{D}} \cdot \mathbf{E}_{\mathcal{D}}) \mathbf{E}_{\mathcal{D}}] d\Sigma + \{\mathcal{D} \rightarrow \mathcal{V}\} + \right. \\ & \quad \left. - \frac{\epsilon - 1}{c} \int_{V_{\mathcal{D}}} \dot{\mathbf{E}} \times \mathbf{B} d\mathbf{r}, \right. \end{aligned} \quad (45)$$

where $\mathbf{E}_{\mathcal{D}}$ and $\mathbf{n}_{\mathcal{D}}$ are the electric field and the external normal respectively, computed on Σ as a limit from the dielectric side and $\{\mathcal{D} \rightarrow \mathcal{V}\}$ represents the same expression, computed as a limit from the vacuum side. There is no term containing the magnetic field, as a consequence of its continuity across Σ , eq.(10).

Using eqs.(9) we get

$$\lim_{\delta \rightarrow 0} \frac{d}{dt} \int_{\mathcal{R}_\infty \ominus \mathcal{L}_\delta} \frac{1}{c} \mathbf{E} \times \mathbf{B} d\mathbf{r} =$$

$$\begin{aligned}
&= \int_{\Sigma} \left[-\frac{1}{2} \mathbf{n}_{\mathcal{D}} \Delta \mathbf{E}^2 + \Delta(E_n \mathbf{E}) \right] d\Sigma - \frac{\epsilon - 1}{c} \int_{V_{\mathcal{D}}} \dot{\mathbf{E}} \times \mathbf{B} d\mathbf{r} = \\
&= -\frac{1}{2} \int_{\Sigma} \tilde{\sigma}_P (\mathbf{E}_{\mathcal{V}} + \mathbf{E}_{\mathcal{D}}) d\Sigma - \frac{1}{c} \int_{V_{\mathcal{D}}} \mathbf{J}_P \times \mathbf{B} d\mathbf{r} \equiv -\tilde{\mathbf{F}}_{\mathcal{D}}(t), \quad (46)
\end{aligned}$$

where

$$\tilde{\mathbf{F}}_{\mathcal{D}}(t) = \frac{1}{2} \int_{\Sigma} \tilde{\sigma}_P (\mathbf{E}_{\mathcal{V}} + \mathbf{E}_{\mathcal{D}}) d\Sigma + \frac{1}{c} \int_{V_{\mathcal{D}}} \mathbf{J}_P \times \mathbf{B} d\mathbf{r} \quad (47)$$

$$= \frac{1}{2} \int_{\Sigma} \tilde{\sigma}_P (\mathbf{E}_{\mathcal{V}} + \mathbf{E}_{\mathcal{D}}) d\Sigma + \frac{\epsilon - 1}{c} \int_{V_{\mathcal{D}}} \dot{\mathbf{E}} \times \mathbf{B} d\mathbf{r}. \quad (48)$$

We now imagine that at an initial time T_1 the radiation is entirely contained inside the vacuum region $V_{\mathcal{V}}$ and at a final time T_2 the radiation is entirely contained inside the dielectric region $V_{\mathcal{D}}$. We are in this way excluding any substantial radiation reflection from the dielectric boundary. While, as we have shown, up to order α this condition is automatically satisfied, this is not assured for the exact solution and we must explicitly assume that the experiment is performed under negligible reflectivity conditions.

Integration of eq.(46) over t from T_1 to T_2 gives

$$\begin{aligned}
\lim_{\delta \rightarrow 0} \int_{\mathcal{R}_{\infty} \ominus \mathcal{L}_{\delta}} \frac{1}{c} \mathbf{E} \times \mathbf{B}|_{T_2} d\mathbf{r} - \lim_{\delta \rightarrow 0} \int_{\mathcal{R}_{\infty} \ominus \mathcal{L}_{\delta}} \frac{1}{c} \mathbf{E} \times \mathbf{B}|_{T_1} d\mathbf{r} = \\
= - \int_{T_1}^{T_2} \tilde{\mathbf{F}}_{\mathcal{D}}(t) dt \quad (49)
\end{aligned}$$

which is interpreted as

$$\mathbf{P}_{\gamma}' - \mathbf{P}_{\gamma} = - \int_{T_1}^{T_2} \tilde{\mathbf{F}}_{\mathcal{D}}(t) dt \equiv -\mathbf{P}_{\mathcal{D}}, \quad (50)$$

suggesting the identification of $\mathbf{P}_{\mathcal{D}}$ with the momentum transferred to \mathcal{D} . This result makes eq.(47) a very natural candidate for the force exerted by the e.m. field on an insulator. Eq.(47) provides the required regularization of the force acting on the dielectric surface Σ : it must be computed using the average of the electric field on Σ from the vacuum and from the dielectric side. This is the natural choice when

computing the force acting on a simple layer charge distribution, as discussed in textbooks[20]. With this prescription eq.(47) represents the precise statement that polarization charges and currents interact in an universal way with the e.m. field.

As a consistency check we notice that $\tilde{\mathbf{F}}_{\mathcal{D}} = 0$ as soon as the e.m. field is completely internal to $V_{\mathcal{D}}$. This is obvious for the first term of eq.(48). As for the second term, $\int_{V_{\mathcal{D}}} \dot{\mathbf{E}} \times \mathbf{B} d\mathbf{r}$, when \mathbf{E} and \mathbf{B} do not touch the boundary¹¹ Σ , can be transformed, using Maxwell equations, as

$$\int_{V_{\mathcal{D}}} \dot{\mathbf{E}} \times \mathbf{B} d\mathbf{r} = \frac{c}{\epsilon} \int_{V_{\mathcal{D}}} (\nabla \times \mathbf{B}) \times \mathbf{B} d\mathbf{r} = 0, \quad (51)$$

through partial integration. Therefore, as soon as the e.m. wave abandons Σ , it ceases to exert any force on the dielectric body, in agreement with what was found in the approximate computation of section 3.1.

These arguments are easily extended to the study of the corresponding energy flow.

The e.m. energy density, \mathcal{U} , for a dielectric material is given by[2]

$$\mathcal{U} = \frac{1}{2}(\epsilon \mathbf{E}^2 + \mathbf{B}^2). \quad (52)$$

In analogy with eq.(44), we compute

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \int_{\mathcal{R}_{\infty} \ominus \mathcal{L}_{\delta}} (\epsilon \mathbf{E}^2 + \mathbf{B}^2) d\mathbf{r} &= \int_{\mathcal{R}_{\infty} \ominus \mathcal{L}_{\delta}} (\epsilon \mathbf{E} \cdot \dot{\mathbf{E}} + \mathbf{B} \cdot \dot{\mathbf{B}}) d\mathbf{r} = \\ &= - \int_{\Sigma} [(\mathbf{E}_{\mathcal{D}} \times \mathbf{B}_{\mathcal{D}}) \cdot \mathbf{n}_{\mathcal{D}} + (\mathbf{E}_{\mathcal{V}} \times \mathbf{B}_{\mathcal{V}}) \cdot \mathbf{n}_{\mathcal{V}}] d\Sigma = \\ &= - \int_{\Sigma} \Delta[(\mathbf{E} \times \mathbf{B}) \cdot \mathbf{n}_{\mathcal{D}}] d\Sigma. \end{aligned} \quad (53)$$

As a consequence of eqs.(9) and (10) we have

$$\Delta[(\mathbf{E} \times \mathbf{B}) \cdot \mathbf{n}_{\mathcal{D}}] = (\Delta \mathbf{E} \times \mathbf{B}) \cdot \mathbf{n}_{\mathcal{D}} = \tilde{\sigma}(\mathbf{n}_{\mathcal{D}} \times \mathbf{B}) \cdot \mathbf{n}_{\mathcal{D}} = 0, \quad (54)$$

¹¹The important point is that ϵ must not show discontinuities within the region of integration.

so that

$$\lim_{\delta \rightarrow 0} \frac{1}{2} \int_{\mathcal{R}_\infty \ominus \mathcal{L}_\delta} (\epsilon \mathbf{E}^2 + \mathbf{B}^2)|_{T_2} d\mathbf{r} = \lim_{\delta \rightarrow 0} \frac{1}{2} \int_{\mathcal{R}_\infty \ominus \mathcal{L}_\delta} (\epsilon \mathbf{E}^2 + \mathbf{B}^2)|_{T_1} d\mathbf{r}, \quad (55)$$

which shows that, again under the assumption of negligible reflection, there are no energy losses in the radiation during the transition from \mathcal{V} to \mathcal{D} .

5 Conclusions

We presented some elementary checks supporting the validity of the Abraham expression of the momentum to be attributed to an e.m. wave packet, in order to obey momentum conservation. We used the approximation $\epsilon \approx 1$, first proposed in ref.[15], which allowed the analytic treatment of some problems. In this way it has been possible to derive the Snell's refraction law through momentum conservation.

In order to overcome the limitations due to the expansion in powers of the polarizability, in section 4 we presented a general, nonperturbative argument showing that Abraham momentum conservation implies that an e.m. field interacts with polarization charges and currents in the same way as it does with the so called *free* ones. The argument shows the correct way of dealing with the discontinuity of the electric field across the dielectric boundary.

Acknowledgements

It is a pleasure to thank professor Giancarlo Ruocco for introducing me to the Abraham-Minkowski controversy and professor Giancarlo Rossi for discussions on the subject.

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